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THE DUAL OF THE BERGMAN SPACE A^1 IN SYMMETRIC SIEGEL DOMAINS OF TYPE II

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ABSTRACT. An affirmative answer is given to the following conjecture of R. Coifman and R. Rochberg: in any symmetric Siegel domain of type II, the dual of the Bergman space A^1 coincides with the Bloch space of holomorphic functions and can be realized as the Bergman projection of L^∞.

Let D be a symmetric Siegel domain of type II; let v denote the Lebesgue measure in D and $H(D)$ the space of holomorphic functions in D. The Bergman space $A^p(D)$, $0 < p \leqslant \infty$, is defined by $A^p(D) = H(D) \cap L^p(dv)$.

In the one-dimensional case, $D = \Pi^+ = \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$, R. Coifman and R. Rochberg [4] gave a proof of the following fact: the dual of the Bergman space $A^1(\Pi^+)$ coincides with the Bloch space of holomorphic functions and can be realized as the Bergman projection of L^∞. Then, these authors conjectured that this characterization of the dual of A^1 should hold in any symmetric Siegel domain of type II.

In [1] (resp. [2 and 3]), an affirmative answer to this conjecture was given in the particular case of the Cayley transform of the unit ball in $\mathbf{C}^n$, $n > 1$, defined by $\{(z_1, z') \in \mathbf{C} \times \mathbf{C}^{n-1}\colon \operatorname{Im} z_1 > |z'|^2\}$ (resp. in the tube $\mathbf{R}^{n+1} + i\Gamma$, $n \geqslant 1$, over the spherical cone Γ defined in $\mathbf{R}^{n+1}$ by

$$\Gamma = \left\{(y_0, y_1, y_2, \ldots, y_n) \in \mathbf{R}^{n+1}\colon y_0 y_1 - y_2^2 - \cdots - y_n^2 > 0, y_0 > 0\right\}\Big).$$

The purpose of this paper is to prove the Coifman-Rochberg conjecture in the general case of any symmetric Siegel domain of type II.

We shall denote by $B(\zeta, z)$ the Bergman kernel of such a domain D. When r is a strictly positive integer, S. G. Gindikin [5] defined a differential operator $\mathscr{D}_r$ in D that satisfies the property

$$(\mathscr{D}_r)_\zeta B(\zeta, z) = C_r B^{1+r}(\zeta, z), \qquad \zeta, z \in D.$$

The Bloch space $\mathscr{B}_r$ corresponding to the integer r is defined in terms of that operator $\mathscr{D}_r$. A holomorphic function g in D is said to be a Bloch function when it satisfies the estimate

$$\sup_{z \in D} \left\{|\mathscr{D}_r g(z)| B^{-r}(z, z)\right\} < +\infty.$$

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Denote by $\mathcal{N}$ the subspace of $H(D)$ consisting of those functions satisfying $\mathcal{D}_r g = 0$; the Bloch space $\mathcal{B}_r$ is defined as the quotient space $\{\text{Bloch functions}\}/\mathcal{N}$.

In view of the proof of the Coifman-Rochberg conjecture, let us recall at once that, unlike the bounded domains, the Bergman kernel $B(\zeta, z)$ of D is not integrable with respect to z because of its bad behaviour when z tends to infinity.

In the first chapter of this paper, we use S. G. Gindikin's general theory [**5**] to characterize those integers r for which the function $z \mapsto B^{1+r}(\zeta, z)$ belongs to $L^1(D)$, $\zeta \in D$; also enclosed in this chapter are some results that will be useful later.

In the second chapter, we prove that when r is sufficiently large, the dual of A^1 coincides with the Bloch space $\mathcal{B}_r$. More precisely, let L be a bounded linear functional on $A^1(D)$; by the Hahn-Banach theorem, there exists a bounded function b in D such that for any f in $A^1(D)$, the following equality holds: $L(f) = \int_D b\bar{f}\,dv$.

We associate with b the holomorphic function G defined in D by

$$G(\zeta) = C_r \int_D B^{1+r}(\zeta, z) b(z)\, dv(z);$$

this function G possesses the following two properties:

1° $$\sup_{\zeta \in D} \left\{ |G(\zeta)| B^{-r}(\zeta, \zeta) \right\} < +\infty;$$

2° $$L(f) = \int_D G(\zeta) \bar{f}(\zeta) B^{-r}(\zeta, \zeta)\, dv(\zeta)$$

for any f in $A^1(D)$.

Since it is well known (cf. [7]) that the differential equation $\mathcal{D}_r \tilde{g} = G$ in D possesses holomorphic solutions, we then conclude that L can be represented by the element g of $\mathcal{B}_r$ consisting of all the holomorphic solutions to this equation.

In the third chapter, we denote by P the operator that assigns to a bounded function b the above-defined element g of the Bloch space $\mathcal{B}_r$, and we call it "Bergman projection" of L^∞ into $\mathcal{B}_r$ for the following reason: although P is not the integral operator $\mathcal{P}$ associated with the Bergman kernel $B(\zeta, z)$ of D, which has no meaning on L^∞, we shall prove that for any function b in $(L^2 \cap L^\infty)(D)$, the element Pb of $\mathcal{B}_r$ can be represented by the holomorphic function $\mathcal{P}b$. Moreover, we prove that P defines a bounded operator from L^∞ onto $\mathcal{B}_r$, and, consequently, the dual of $A^1(D)$ (which coincides with the Bloch space $\mathcal{B}_r$) can be realized as the Bergman projection of L^∞.

Finally, as usual, the same letter C will denote constants that may be different from each other.

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I. Preliminary results about the Bergman kernel

1. The canonical decomposition of the domain. Let V denote an affine-homogeneous cone in $\mathbf{R}^n$, which contains no straight line; we first recall the canonical decomposition of V as settled in [**5**].

NOTATIONS. (i) At the jth step, $j = 1, 2, \ldots$, the real line $\mathbf{R}$ will be denoted by $\mathbf{R}^1_{jj}$ and at the kth step, $k = 2, 3, \ldots$, $\mathbf{R}^{n_k}_k$ will denote the n_k-dimensional euclidean space $\mathbf{R}^{n_k}$.

(ii) Let P denote the real Siegel domain

$$P = \{(y, t) \in \mathbf{R}^\sigma \times \mathbf{R}^\tau : y - \varphi(t, t) \in \Gamma\}$$

where $\Gamma \subset \mathbf{R}^\sigma$ is an affine-homogeneous cone which contains no straight line and φ is a Γ-bilinear symmetric form defined in $\mathbf{R}^\tau$; we shall denote by $V(P)$ the cone

$$V(P) = \{(y, t, r) \in \mathbf{R}^\sigma \times \mathbf{R}^\tau \times \mathbf{R} : r > 0 \text{ and } (yr, t) \in P\}.$$

In order to obtain the canonical decomposition of the cone V, we consider at the first step the cone $V^{(1)} = (0, \infty) \subset \mathbf{R}^1_{11}$.

At the second step, we associate with the cone $V^{(1)}$ and with a $V^{(1)}$-bilinear symmetric form $F^{(2)}$ defined in $\mathbf{R}^{n_2}_2$ a real Siegel domain $P^{(2)}$ contained in $\mathbf{R}^1_{11} \times \mathbf{R}^{n_2}_2$ and the cone

$$V^{(2)} = V(P^{(2)}) \subset \mathbf{R}^1_{11} \times \mathbf{R}^{n_2}_2 \times \mathbf{R}^1_{22}.$$

At the third step, we associate with the cone $V^{(2)}$ and with a $V^{(2)}$-bilinear symmetric form $F^{(3)}$ defined in $\mathbf{R}^{n_3}_3$ a real Siegel domain $P^{(3)}$ contained in $\mathbf{R}^1_{11} \times \mathbf{R}^{n_2}_2 \times \mathbf{R}^1_{22} \times \mathbf{R}^{n_3}_3$ and the cone

$$V^{(3)} = V(P^{(3)}) \subset \mathbf{R}^1_{11} \times \mathbf{R}^{n_2}_2 \times \mathbf{R}^1_{22} \times \mathbf{R}^{n_3}_3 \times \mathbf{R}^1_{33}.$$

And so on, at the ith step, we associate with the cone $V^{(i-1)}$ and with a $V^{(i-1)}$-bilinear symmetric form $F^{(i)}$ defined in $\mathbf{R}^{n_i}_i$ a real Siegel domain $P^{(i)}$ and the cone $V^{(i)} = V(P^{(i)})$.

It follows from results in [5] that every affine-homogeneous cone V, which contains no straight line, can be decomposed in the form $V = V^{(i)}$ (up to an affine isomorphism). The required number of steps to obtain V in this form is called the rank l of the cone V $(V = V^{(l)})$. This yields the following decomposition of the space $\mathbf{R}^n$ containing V:

$$\mathbf{R}^n = \mathbf{R}^1_{11} \times \cdots \times \mathbf{R}^1_{ll} \times \mathbf{R}^{n_2}_2 \times \cdots \times \mathbf{R}^{n_l}_l, \qquad n = l + \sum_{j=2}^{l} n_j.$$

Now, let $F^{(j)}_{ii}$, $1 \leqslant i < j \leqslant l$, denote the projection of the $V^{(j-1)}$-bilinear symmetric form $F^{(j)}$ on the real line $\mathbf{R}^1_{ii}$, and let $\mathbf{R}^{n_{ij}}_{ij}$ denote the n_{ij}-dimensional subspace of $\mathbf{R}^{n_j}_j$ where the form $F^{(j)}_{ii}$ is positive definite; then the decomposition $\mathbf{R}^{n_j}_j = \prod_{i=1}^{j-1} \mathbf{R}^{n_{ij}}_{ij}$ holds.

We recall next that the cone V is self-conjugate if and only if the integers n_{ij}, $1 \leqslant i < j \leqslant l$, are equal among themselves. In that case, when x_{ij} denotes the projection of $x \in \mathbf{R}^n$ on $\mathbf{R}^{n_{ij}}_{ij}$, the cone V is self-conjugate with respect to the scalar product

$$\langle x, x' \rangle = \sum_{j=1}^{l} \frac{(x_{jj} x'_{jj})}{2} + \sum_{1 \leqslant i \leqslant j \leqslant l} x_{ij} x'_{ij}, \qquad x, x' \in \mathbf{R}^n.$$

Thus, in order that the cone $V \subset \mathbf{R}^n$ of rank l be self-conjugate, we shall assume that $n - l$ is a multiple in $\mathbf{N}$ of $l(l-1)/2$ and that all the n_{ij}'s, $1 \leqslant i < j \leqslant l$, are equal to $p = 2(n-l)/l(l-1)$.

We shall also assume that the vectors in the subspaces $\mathbf{R}^p_{ij}$ are represented by their coordinates with respect to bases satisfying the following property: for any λ_j in $R^{n_j}_j$, one has the equality $F^{(j)}_{ii}(\lambda_j, \lambda_j) = \|\lambda_{ij}\|^2$, where $\|\cdot\|$ denotes the euclidean norm in $\mathbf{R}^p_{ij}$.

Furthermore, let $F^{(k)}_{ij}$, $1 \leqslant i < j < k \leqslant l$, denote the projection of the $V^{(k-1)}$-bilinear symmetric form $F^{(k)}$ on $\mathbf{R}^p_{ij}$; the homogeneity assumption on the cone V implies that the form $F^{(k)}_{ij}$ is concentrated on $\mathbf{R}^p_{ik} \times \mathbf{R}^p_{jk}$.

Now, let $D \subset \mathbf{C}^n \times \mathbf{C}^N$ denote a symmetric Siegel domain of type II, associated to the cone V and to the V-Hermitian form F. We next describe the canonical decomposition of the space $\mathbf{C}^n \times \mathbf{C}^N$ containing D.

First, the decomposition

$$\mathbf{R}^n = \prod_{j=1}^{l} \mathbf{R}^1_{jj} \times \prod_{1 \leqslant j < k \leqslant l} \mathbf{R}^p_{jk}$$

given above yields in a natural way the decomposition

$$\mathbf{C}^n = \prod_{j=1}^{l} \mathbf{C}^1_{jj} \times \prod_{1 \leqslant j < k \leqslant l} \mathbf{C}^p_{jk}.$$

Secondly, let F_{jj} denote the projection of the form F on the complex plane $\mathbf{C}^1_{jj}$, and let $\mathbf{C}^{q_j}_j$ denote the q_j-dimensional subspace of $\mathbf{C}^N$ where the form F_{jj} is positive definite; the decomposition $\mathbf{C}^N = \prod_{j=1}^{l} \mathbf{C}^{q_j}_j$ holds, and the canonical decomposition of the space $\mathbf{C}^n \times \mathbf{C}^N$ containing D is given by

$$\mathbf{C}^n \times \mathbf{C}^N = \prod_{j=1}^{l} \mathbf{C}^1_{jj} \times \prod_{1 \leqslant j < k \leqslant l} \mathbf{C}^p_{jk} \times \prod_{j=1}^{l} \mathbf{C}^{q_j}_j.$$

Moreover, with respect to its canonical form, the cone V can be described in the following quantitative manner. Let λ be a point of V and let λ_j, $j = 2, \ldots, l$, denote the projection of λ on $\mathbf{R}^{n_j}_j$; there exists an automorphism (in the transitive group of V given in [5]) that assigns to λ a point $\tilde{\lambda}$ of V satisfying $\tilde{\lambda}_j = 0$, for every $j = 2, \ldots, l$. Such an automorphism can be built in $l-1$ stages: a first automorphism maps λ to $\lambda^{(1)} \in V$ satisfying $\lambda^{(1)}_l = 0$, a second one takes $\lambda^{(1)}$ to $\lambda^{(2)} \in V$ satisfying $\lambda^{(2)}_{l-1} = \lambda^{(2)}_l = 0, \ldots,$ and finally, the $(l-1)$th automorphism assigns to $\lambda^{(l-2)}$ satisfying $\lambda^{(l-2)}_3 = \cdots = \lambda^{(l-2)}_l = 0$ the point $\lambda^{(l-1)} = \tilde{\lambda}$. More explicitly, the points $\lambda^{(m)}$, $m = 0, 1, \ldots, l-1$, of V are given by the following recurrence formulas (formulas (1.26), (1.27) in [5]): $\lambda^{(0)} = \lambda$,

$$\lambda^{(m+1)}_{ij} = \lambda^{(m)}_{ij} - \frac{F^{(l-m)}_{ij}\left(\lambda^{(m)}_{l-m}, \lambda^{(m)}_{l-m}\right)}{\lambda^{(m)}_{l-m,l-m}}, \qquad 1 \leqslant i \leqslant j < l - m.$$

Now, set $\chi_j(\lambda) = \lambda^{(l-j)}_{jj}$; the cone V is defined by the l inequalities $\chi_j(\lambda) > 0$, $j = 1, 2, \ldots, l$ (formula (1.25) in [5]).

Furthermore, let us recall that the functions $\chi_j(\lambda)$ can also be obtained in the following way. The transitive group $G(V)$ of the cone V, defined in [5], is represented by $l \times l$ triangular matrices, and the automorphism $T \in G(V)$ that assigns $\tilde{\lambda}$ to λ is a triangular matrix of $G(V)$ whose diagonal elements are all equal to 1. On the other hand, let e denote the point of $V \subset \prod_{j=1}^{l} \mathbf{R}^1_{jj} \times \prod_{j=2}^{l} \mathbf{R}^{n_j}_j$ whose coordinates are $e_{jj} = 1$ and $e_k = 0$ for $j = 1, \ldots, l$ and $k = 2, \ldots, l$; then for any λ in V, there exists in $G(V)$ a unique matrix g such that $g(e) = \lambda$ and g^{-1} is the product of T by a diagonal matrix. Hence, the matrix g can be written as the product $g = td$ of a diagonal matrix d of $G(V)$ by a triangular matrix t of $G(V)$ whose diagonal elements are all equal to 1; the point $\tilde{\lambda}$ and the functions $\chi_j(\lambda)$ are then given by the formulas $\tilde{\lambda} = d(e)$ and $\chi_j(\lambda) = d_j$, where d_j is the jth diagonal element of d.

Finally, let us recall that the self-conjugate cone V can also be defined by the l functions $\chi_j^*(\lambda)$ that generally define the dual cone V^* of V. Those functions $\chi_j^*(\lambda)$ are given as follows.

Let $G^*(V)$ denote the adjoint group of $G(V)$ with respect to the scalar product $\langle\ ,\ \rangle$; the group $G^*(V)$ acts transitively on $V^* = \overline{V}$ and for any λ in $V^* = \overline{V}$, there exists a unique automorphism g^* in $G^*(V)$ satisfying $g^*(e) = \lambda$. The group $G^*(V)$ can also be represented by triangular matrices, and the matrix g^* can be uniquely written in $G^*(V)$ as the product $g^* = t^*d^*$ of a diagonal matrix d^* by a triangular matrix t^* whose diagonal elements are all equal to 1. The functions $\chi_j^*(\lambda)$ are then given by $\chi_j^*(\lambda) = d_j^*$, where d_j^* is the jth diagonal element of d^*.

We shall carry the following notations: for any vector $\rho = (\rho_1, \ldots, \rho_l)$ in R^l, we set

$$\lambda^\rho = \prod_{j=1}^{l} \left(\chi_j(\lambda)\right)^{\rho_j} \quad \text{and} \quad (\lambda^*)^\rho = \prod_{j=1}^{l} \left(\chi_j^*(\lambda)\right)^{\rho_j}.$$

2. The Bergman kernel. Let n and l be two integers satisfying $1 \leqslant l \leqslant n$, $n > 1$, and such that $l(l-1)/2$ divides $n - l$ in $\mathbf{N}$. By V we shall denote an affine-homogeneous, self-conjugate cone of rank l in $\mathbf{R}^n$, and we shall assume that the cone V is irreducible: this implies that $1 < l < n$.

We therefore exclude the case $n = l = 1$, where the cone V is the positive real half-line and its associated Siegel domains, that are Cayley transforms of balls, were studied in [1].

Let D denote a symmetric Siegel domain of type II contained in $\mathbf{C}^n \times \mathbf{C}^N$, associated to the cone V and to the V-Hermitian form F defined in $\mathbf{C}^N$; the space $\mathbf{C}^n \times \mathbf{C}^N$ containing D will be considered in its canonical form

$$\mathbf{C}^n = \prod_{j=1}^{l} \mathbf{C}^1_{jj} \times \prod_{k=2}^{l} \mathbf{C}^{n_k}_k,$$

$$\mathbf{C}^{n_k}_k = \prod_{j<k} \mathbf{C}^p_{jk}, \quad p = \frac{2(n-l)}{l(l-1)},$$

$$\mathbf{C}^N = \prod_{j=1}^{l} \mathbf{C}^{q_j}_j.$$

We shall denote by q the vector of $\mathbf{N}^l$ whose coordinates are q_j, $j = 1, \dots, l$, and we shall denote by d and m the vectors of $\mathbf{R}^l$ whose respective coordinates are $d_j = -(1 + p(l-1)/2)$, $m_j = p(j-1)$.

It then follows from results in [5] that the Bergman kernel $B(\zeta, z)$ of D has the following two expressions:

PROPOSITION I.2.1. *The Bergman kernel $B(\zeta, z)$ of D is given by the formulas*

$$(1)\qquad B(\zeta, z) = c\left(\frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\zeta_2, z_2)\right)^{2d-q}$$

$$(2)\qquad = c'\int_V \exp\left(-\left\langle \lambda, \frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\zeta_2, z_2)\right\rangle\right)(\lambda^*)^{-d+q}\, d\lambda,$$

where $\zeta = (\xi + i\eta, \zeta_2)$ and $z = (x + iy, z_2)$ are two points of $D \subset \mathbf{C}^n \times \mathbf{C}^N$.

Subsequently, we shall carry the following notations:

(i) let $\rho = (\rho_1, \dots, \rho_l)$ and $\rho' = (\rho'_1, \dots, \rho'_l)$ be two vectors in $\mathbf{R}^l$; we shall write $\rho > \rho'$ whenever $\rho_j > \rho'_j$ for every $j = 1, \dots, l$; we shall also set

$$\rho\rho' = (\rho_1\rho'_1, \dots, \rho_l\rho'_l) \quad \text{and} \quad \rho/\rho' = (\rho_1/\rho'_1, \dots, \rho_l/\rho'_l),$$

when none of the ρ'_j's, $j = 1, \dots, l$, is zero;

(ii) for any two points $\zeta = (\xi + i\eta, \zeta_2)$ and $z = (x + iy, z_2)$ in $D \subset \mathbf{C}^n \times \mathbf{C}^N$, we let $B^\rho(z, z)$ and $B^{1+\rho}(\zeta, z)$ denote the expressions

$$B^\rho(z, z) = c_\rho\left(y - F(z_2, z_2)\right)^{(2d-q)\rho}$$

and

$$B^{1+\rho}(\zeta, z) = c'_\rho\left(\frac{\xi - x}{2i} + \frac{\eta + y}{2} - F(\zeta_2, z_2)\right)^{2d-q+(2d-q)\rho}.$$

We will use the following lemma, due to S. G. Gindikin [5]:

LEMMA I.2.1. *A holomorphic function G in D belongs to $A^2(D)$ if and only if it can be expressed in the form*

$$G(z) = \int_V g(\lambda, z_2)\exp(i\langle \lambda, z_1\rangle)\, d\lambda,$$

for any $z = (z_1, z_2)$ in $D \subset \mathbf{C}^n \times \mathbf{C}^N$, where the function $g(\lambda, z_2)$ satisfies the following properties:

(i) *the function $z_2 \mapsto g(\lambda, z_2)$ is entire in $\mathbf{C}^N$*;

(ii) *the integral*

$$\int_V \int_{\mathbf{C}^N} |g(\lambda, z_2)|^2 \exp(-2\langle \lambda, F(z_2, z_2)\rangle)(\lambda^*)^d\, dv(z_2)\, d\lambda$$

converges and is equal to $c\int_D |G(z)|^2\, dv(z)$.

We state another lemma of Gindikin [5]:

LEMMA I.2.2. *For any vector ρ in $\mathbf{R}^l$ such that $\rho > m/2$ and for any z in $\mathbf{C}^n$ such that* $\operatorname{Re} z \in V$, *we have*

$$\int_V \exp(-\langle z, \lambda\rangle)(\lambda^*)^{\rho + d}\, d\lambda = c_\rho (z)^{-\rho}.$$

Our next purpose is to prove the following estimate for the Bergman kernel:

LEMMA I.2.3. *For any ζ in D, the kernel $B^{1+\alpha}(\zeta, z)$, $\alpha \in \mathbf{R}^l$, belongs to $L^1(dv(z))$ if and only if $(-2d+q)\alpha > m/2$; in that case,*

$$\int_D |B^{1+\alpha}(\zeta, z)|\, dv(z) = c_\alpha B^\alpha(\zeta, \zeta).$$

PROOF. Because of the homogeneity of the domain D, it suffices to prove the lemma for $\zeta = (ie, 0) \in D \subset \mathbf{C}^n \times \mathbf{C}^N$, where e is the point of $V \subset \prod_{j=1}^l \mathbf{R}^1_{jj} \times \prod_{1 \leqslant j < k \leqslant l} \mathbf{R}^p_{jk}$ whose components are $e_{jj} = 1$ for $j = 1, \ldots, l$ and $e_{jk} = 0$ when $j < k$.

Take then $\zeta = (ie, 0)$. The assumption $(-2d+q)\alpha > m/2$ yields the inequality $-[2d - q + (2d-q)\alpha]/2 > m/2$; now, in view of Proposition I.2.1 and Lemma I.2.2, $B^{(1+\alpha)/2}(\zeta, z)$ can be written.

$$\begin{aligned} B^{(1+\alpha)/2}(\zeta, z) &= c_\alpha \left(-\frac{x}{2i} + \frac{e+y}{2} \right)^{[2d-q+(2d-q)\alpha]/2} \\ &= c'_\alpha \int_V \exp\left(-\left\langle \lambda, -\frac{x}{2i} + \frac{e+y}{2} \right\rangle \right) (\lambda^*)^{[q+(-2d+q)\alpha]/2}\, d\lambda. \end{aligned}$$

Next, the Plancherel formula given in Lemma I.2.1 yields

$$\begin{aligned} \int_D |B^{1+\alpha}(\zeta, z)|\, dv(z) &= c_\alpha \int_V \exp(-\langle \lambda, e \rangle)(\lambda^*)^{d+q+(-2d+q)\alpha} \\ &\quad \cdot \left(\int_{\mathbf{C}^N} \exp(-2\langle \lambda, F(z_2, z_2) \rangle)\, dv(z_2) \right) d\lambda. \end{aligned}$$

Let us show that the right-hand side of this last equality converges if $(-2d+q)\alpha > m/2$. We first integrate with respect to z_2; the following lemma is proved in [**5**]:

LEMMA I.2.4. *The integral*

$$\int_{\mathbf{C}^N} \exp(-\langle e, F(u, u) \rangle)\, dv(u)$$

is convergent.

In view of lemma I.2.4 and of the homogeneity of the cone V, we get

$$\int_{\mathbf{C}^N} \exp(-2\langle \lambda, F(z_2, z_2) \rangle)\, dv(z_2) = c(\lambda^*)^{-q};$$

hence,

$$\int_D |B^{1+\alpha}((ie, 0), z)|\, dv(z) = c_\alpha \int_V \exp(-\langle \lambda, e \rangle)(\lambda^*)^{d+(-2d+q)\alpha}\, d\lambda. \tag{3}$$

It then follows from Lemma I.2.2 that $B^{1+\alpha}((ie, 0), z)$ belongs to $L^1(dv(z))$ when $(-2d+q)\alpha > m/2$.

Conversely, S. G. Gindikin [**5**] has proved that the right-hand side of (3) converges only if this condition on α is satisfied. The proof of the lemma is complete.

REMARK. The necessary and sufficient condition on α given in Lemma I.2.3 contradicts a statement of R. Coifman and R. Rochberg (Lemma 2.2 of [**4**]). Even in the particular case of the tube over the spherical cone of $\mathbf{R}^n$, $n \geqslant 3$ (cone of rank two), the condition $\alpha > 0$ given by those authors was proved before to be insufficient (cf. [**2** and **3**]).

We shall also use Lemma 2.3 of [**4**]:

LEMMA I.2.5. *d denotes the Bergman distance in D. There exists a constant C_D such that for any two points ζ and ζ_0 in D satisfying $d(\zeta, \zeta_0) < 10$ and for any z in D, the following inequality holds*:

$$|B(\zeta, z)/B(\zeta_0, z) - 1| \leqslant c_D d(\zeta, \zeta_0).$$

II. THE DUAL OF A^1 AND BLOCH FUNCTIONS

In the following, ρ denotes a vector of $\mathbf{N}^l$ whose coordinates ρ_j, $j = 1, \ldots, l$, satisfy two conditions:

(i) $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_{l-1} \geqslant \rho_l = 1$;

(ii) $(\lambda^*)^\rho = \prod_{j=1}^l \chi_j^{*\rho_j}$ is a polynomial of $\lambda \in \mathbf{R}^n$;

according to the terminology in [**5**], a vector ρ satisfying property (ii) is said to be V-integral, and for any cone V of rank l an example of a V-integral vector is $\rho = (2^{l-2}, 2^{l-3}, \ldots, 2, 1, 1)$.

We shall denote by $\mathscr{R}$ the differential polynomial in $\mathbf{C}^n$ that possesses the property

$$\mathscr{R}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^\rho \exp(\langle \lambda, z_1 \rangle), \qquad z_1 \in \mathbf{C}^n. \tag{4}$$

Referring to [**5**], $\mathscr{R}$ is a Riemann-Liouville differential operator of $D \subset \mathbf{C}^n \times \mathbf{C}^N$.

Let r be a vector of $\mathbf{N}^l$ satisfying the following two conditions (the notations are those introduced in section I.2):

(iii) $(\lambda^*)^{r\rho}$ is a polynomial of $\lambda \in \mathbf{R}^n$; i.e., $r\rho$ is also a V-integral vector;

(iv) $r\rho > m/2$.

Let $\mathscr{D} = \mathscr{R}^r$ denote the differential polynomial in $\mathbf{C}^n$ that possesses the following property:

$$\mathscr{D}_{z_1} \exp(\langle \lambda, z_1 \rangle) = (\lambda^*)^{r\rho} \exp(\langle \lambda, z_1 \rangle), \qquad z_1 \in \mathbf{C}^n.$$

In other words, $\mathscr{D}$ is the Riemann-Liouville differential operator of D obtained by iterating $\mathscr{R}$ r times.

Now, fix two such vectors ρ and r; the corresponding Bloch space $\mathscr{B} = \mathscr{B}_{\rho,r}$ of D is defined as follows. A holomorphic function g in D is said to be a Bloch function if it satisfies

$$\|g\|_* = \sup_{(z_1, z_2) \in D \subset \mathbf{C}^n \times \mathbf{C}^N} \left\{ |\mathscr{D}_{z_1} g(z)| B^{-r\rho/(-2d+q)}(z, z) \right\} < +\infty.$$

Let $\mathscr{N} = \{g \in H(D)\colon \mathscr{D}_{z_1} g(z) = 0\}$; the Bloch space $\mathscr{B} = \mathscr{B}_{\rho,r}$ of D is the quotient space $\{\text{Bloch functions}\}/\mathscr{N}$.

The Bloch space $\mathscr{B} = \mathscr{B}_{\rho,r}$ possesses the following property:

PROPOSITION II.1. *Under the quotient norm induced by $\| \quad \|_*$, the Bloch space $\mathscr{B} = \mathscr{B}_{\rho,r}$ of D is a Banach space.*

The proof of this proposition essentially relies on the following lemma [7, p. 477]:

LEMMA II.1. *For any holomorphic function* $h(z_1, z_2)$ *in* $D \subset \mathbf{C}^n \subset \mathbf{C}^N$, *there exists a holomorphic function* $f(z_1, z_2)$ *such that* $\mathscr{D}_{z_1} f = h$.

We next prove the following theorem:

THEOREM II.1. *For any vectors* ρ *and* r *in* $\mathbf{N}^l$ *satisfying conditions* (i)–(iv), *the dual of the Bergman space* $A^1(D)$ *coincides with the Bloch space* $\mathscr{B} = \mathscr{B}_{\rho,r}$ *of* D.

PROOF. It is easy to prove that the Bloch space $\mathscr{B}_{\rho,r}$ of D coincides with a subspace of the dual of $A^1(D)$. This is done as follows: any element g of $\mathscr{B}_{\rho,r}$ defines an element L of the dual of $A^1(D)$ by

$$L(f) = \int_D \mathscr{D}_{z_1} g(z) \bar{f}(z) B^{-r\rho/(-2d+q)}(z, z)\, dv(z), \qquad f \in A^1(D)$$

and the conclusion follows from the easily proved inequality $\|L\| \leqslant \|g\|_*$.

Conversely, let us prove that any element L of the dual of $A^1(D)$ can be represented by an element g of $\mathscr{B}_{\rho,r}$. First, by the Hahn-Banach theorem, there exists a bounded function b in D such that for any f in $A^1(D)$, $L(f) = \int_D b\bar{f}\, dv$.

Secondly, in view of the condition $r\rho > m/2$ and of Lemma I.2.3, we can assign to this function b the holomorphic function G defined in D by

$$G(\zeta) = \int_D B^{1 + r\rho/(-2d+q)}(\zeta, z) b(z)\, dv(z);$$

G satisfies the estimate

$$(*) \qquad \sup_{\zeta \in D} \left\{ |G(\zeta)| B^{-r\rho/(-2d+q)}(\zeta, \zeta) \right\} \leqslant C\|b\|_\infty$$

and yields an element L' of the dual of $A^1(D)$ defined by

$$L'(f) = C\int_D G(\zeta) \bar{f}(\zeta) B^{-r\rho/(-2d+q)}(\zeta, \zeta)\, dv(\zeta).$$

Now, it follows from the Fubini theorem that

$$L'(f) = C\int_D \left(\int_D B^{1 + r\rho/(-2d+q)}(\zeta, z) \bar{f}(\zeta) B^{-r\rho/(-2d+q)}(\zeta, \zeta)\, dv(\zeta) \right) b(z)\, dv(z).$$

We then conclude, in view of the reproducing formula

$$\int_D B^{1 + r\rho/(-2d+q)}(\zeta, z) \bar{f}(\zeta) B^{-r\rho/(-2d+q)}(\zeta, \zeta)\, dv(\zeta) = C^{-1}\bar{f}(z),$$

that the linear functionals L and L' coincide on $A^1(D)$.

Finally, the linear functional L on $A^1(D)$ can be represented by the element g of the Bloch space $\mathscr{B} = \mathscr{B}_{\rho,r}$ defined as follows: by Lemma II.1 and using the estimate $(*)$ for G, take g to be the equivalence class of all holomorphic solutions to the equation

$$\mathscr{D}_{z_1} g(z) = CG(z), \qquad z = (z_1, z_2) \in D \subset \mathbf{C}^n \times \mathbf{C}^N.$$

Also, $\|g\|_* \leqslant C\|b\|_\infty$. The proof of Theorem II.1 is complete.

REMARK. Bloch spaces can be defined exactly in the same way on homogeneous but nonsymmetric Siegel domains of type II associated with a self-conjugate cone. Furthermore, the argument in the proof of Theorem II.1 leads to the same conclusion in such a domain because all results we need from the first chapter are still valid. Concerning the existence of such domains, let us recall that the "historical" example of a homogeneous but nonsymmetric Siegel domain of type II, due to Pyateckii-Shapiro (in his solution to E. Cartan's problem about the existence of bounded homogeneous but nonsymmetric domains; cf. [**8**, p. 269]) is associated with the (self-conjugate) spherical cone of $\mathbf{R}^3$.

III. THE BERGMAN PROJECTION OF L^∞.

In this chapter, D again denotes a symmetric Siegel domain of type II contained in $\mathbf{C}^n \times \mathbf{C}^N$. The Bergman projection $\mathscr{P}$ of D is the orthogonal projection of $L^2(dv)$ onto the subspace $A^2(D)$ of $L^2(dv)$ consisting of holomorphic functions; for any φ in $L^2(dv)$, $\mathscr{P}\varphi$ is given by the integral formula

$$\mathscr{P}\varphi(\zeta) = \int_D B(\zeta, z)\varphi(z)\, dv(z), \ \zeta \in D, \tag{5}$$

where B denotes the Bergman kernel of D, explicitly given in (1) and (2). However, since by Lemma I.2.3, the kernel $B(\zeta, z)$ does not belong to $L^1(dv(z))$, $\zeta \in D$, expression (5) does not make sense when φ is any bounded function in D.

In the following, we fix two vectors ρ and r in $\mathbf{N}^l$, satisfying conditions (i)–(iv) introduced at the beginning of Chapter II and $\mathscr{B} = \mathscr{B}_{\rho,r}$ denotes the corresponding Bloch space.

Our next concern is to define an operator P from $L^\infty(D)$ into the Bloch space $\mathscr{B}$ that satisfies the following two properties:

1° The image Pb in $\mathscr{B}$ of a function $\varphi \in (L^\infty \cap L^2)(D)$ can be represented by the function $\mathscr{P}\varphi$ defined in (5): for this reason, we shall also call this new operator P the "Bergman projection";

2° The "Bergman projection" P is a bounded operator from L^∞ onto $\mathscr{B}$, and, consequently, the dual of the Bergman space A^1 (the Bloch space $\mathscr{B}$) coincides with the "Bergman projection" PL^∞ of L^∞.

Let b be a bounded function in D. In the proof of Theorem II.1, we assigned to b a holomorphic function G defined in D by

$$G(\zeta) = C\int_D B^{1+r\rho/(-2d+q)}(\zeta, z) b(z)\, dv(z), \qquad \zeta \in D,$$

and the equivalence class g in the Bloch space $\mathscr{B}$, consisting of all holomorphic solutions $\tilde{g}$ of the differential equation

$$\mathscr{D}_{\zeta_1}\tilde{g}(\zeta) = G(\zeta), \qquad \zeta = (\zeta_1, \zeta_2) \in D \subset \mathbf{C}^n \times \mathbf{C}^N.$$

The operator P from L^∞ into $\mathscr{B}$, which assigns g to b, will be called the "Bergman projection" in view of the following lemma:

LEMMA III.1. *For any φ in $(L^\infty \cap L^2)(D)$, the element $P\varphi$ of the Bloch space $\mathscr{B}$ can be represented by the function $\mathscr{P}\varphi$ defined in* (5).

PROOF. This follows immediately from the equality

(6) $$\mathscr{D}_{\zeta_1} B(\zeta, z) = C B^{1+rp/(-2d+q)}(\zeta, z), \qquad \zeta = (\zeta_1, \zeta_2) \in D \subset \mathbf{C}^n \times \mathbf{C}^N.$$

It was already shown that P is a bounded operator from L^∞ into $\mathscr{B}$; we now prove that P is onto. More precisely, we show that any element g of $\mathscr{B}$ is the Bergman projection Pb of a bounded function b defined by

$$b(z) = \mathscr{D}_{z_1} g(z) B^{-rp/(-2d+q)}(z, z), \qquad z = (z_1, z_2) \in D \subset \mathbf{C}^n \times \mathbf{C}^N.$$

This is an immediate consequence of the following proposition:

PROPOSITION III.1. *For any holomorphic function G in D satisfying the estimate*

(*) $$\sup_{z \in D} \left\{ |G(z)| B^{-rp/(-2d+q)}(z, z) \right\} < +\infty,$$

the following equality holds:

(7) $$G(\zeta) = C \int_D G(z) B^{-rp/(-2d+q)}(z, z) B^{1+rp/(-2d+q)}(\zeta, z)\, dv(z).$$

PROOF. This proposition will be proved in an equivalent form on a bounded realization of D. We need the following lemma:

LEMMA III.2. *Every symmetric Siegel domain of type* II *contained in* $\mathbf{C}^M$ *is biholomorphic to a bounded domain* Ω *which possesses the following property*: *if* $z = (z_1, \ldots, z_M)$ *is a point of* Ω, *then for any complex numbers* α_j *satisfying* $|\alpha_j| \leqslant 1$, $j = 1, \ldots, M$, $(\alpha_1 z_1, \ldots, \alpha_M z_M)$ *is also a point of* Ω (Ω *is said to be a bounded "circular" domain*).

Furthermore, let $B_\Omega(\zeta, z)$ *denote the Bergman kernel of* Ω; *there exists a nonzero constant* c_Ω *such that* $B_\Omega(0, z) \equiv C_\Omega$.

The first part of the lemma was proved by A. Koranyi and J. A. Wolff in **[6]**. The second part is contained in the proof of Lemma 2.3 of **[4]** (Lemma I.2.5 above); that proof, due to Koranyi, uses all of Lemma III.2.

We can now begin the proof of Proposition III.1. Let Φ denote a biholomorphic transformation of a bounded circular domain Ω, associated with D by Lemma III.2, onto D, which assigns to the origin 0 the point $e = (e_1, e_2)$ of $D \subset \mathbf{C}^n \times \mathbf{C}^N$, whose components are $(e_1)_{jj} = 1$, $(e_1)_{jk} = 0$ when $j < k$ and $e_2 = 0$.

This transformation Φ yields the following relation between the respective Bergman kernels B_D and B_Ω of the domains D and Ω:

$$B_\Omega(\zeta', z') = B_D(\Phi(\zeta'), \Phi(z'))[J\Phi(\zeta')]\left[\overline{J\Phi(z')}\right],$$

where $J\Phi(w)$ denotes the jacobian of Φ at the point w of Ω. When one puts $z' = 0$ in this relation, it follows from the second part of Lemma III.2 that $J\Phi(\zeta') = C B_D^{-1}(\Phi(\zeta'), e)$; furthermore, according to the notations introduced in section I.2, we set

$$[J\Phi(\zeta')]^\alpha = C_\alpha B_D^{-\alpha}(\Phi(\zeta'), e)$$

and

$$B_\Omega^\alpha(\zeta', z') = B_D^\alpha(\Phi(\zeta'), \Phi(z'))[J\Phi(\zeta')]^\alpha\left[\overline{J\Phi(z')}\right]^\alpha.$$

By means of the biholomorphic transformation $\Psi = \Phi^{-1}$, it is then easy to check that Proposition III.1 is equivalent to the following result:

PROPOSITION III.2. *For any holomorphic function $\tilde{G}$ in Ω satisfying the estimate*

$$\sup_{z'\in\Omega}\left\{|\tilde{G}(z')|B_\Omega^{-r\rho/(-2d+q)}(z',z')|J\Phi(z')|^{2r\rho/(-2d+q)}\right\} < +\infty,$$

the following equality holds:

$$\tilde{G}(\zeta')[J\Phi(\zeta')]^{1+r\rho/(-2d+q)} = C\int_\Omega \tilde{G}(z')[J\Phi(z')]^{1+r\rho/(-2d+q)} \cdot B_\Omega^{1+r\rho/(-2d+q)}(\zeta',z')B_\Omega^{-r\rho/(-2d+q)}(z',z')\,dv(z'), \qquad \zeta'\in\Omega.$$

PROOF OF PROPOSITION III.2. With respect to the measure $B_D^{-r\rho/(-2d+q)}(z,z)\,dv(z)$ the Bergman kernel of D is $CB_D^{1+r\rho/(-2d+q)}(\zeta,z)$. Carrying ourselves onto Ω by using the biholomorphic transformation $\Psi = \Phi^{-1}$, we easily deduce that with respect to the measure $B_\Omega^{-r\rho/(-2d+q)}(z',z')\,dv(z')$, the Bergman kernel of Ω is $CB^{1+r\rho/(-2d+q)}(\zeta',z')$.

It now suffices to prove that the function $\tilde{G}(z')[J\Phi(z')]^{1+r\rho/(-2d+q)}$ is integrable in D with respect to the measure $B_\Omega^{-r\rho/(-2d+q)}(z',z')\,dv(z')$.

Since we assume that the function $\tilde{G}(z')[J\Phi(z')]^{2r\rho/(-2d+q)}B_\Omega^{-r\rho/(-2d+q)}(z',z')$ is bounded in Ω, it is enough to prove the following lemma:

LEMMA III.3. $\int_\Omega|J\Phi(z')|^{1-r\rho/(-2d+q)}\,dv(z') < +\infty$.

PROOF OF LEMMA III.3. Put $\Psi = \Phi^{-1}$; the assertion in the lemma is equivalent in D to the estimate

$$\int_D |J\Psi(z)|^{1+r\rho/(-2d+q)}\,dv(z) < +\infty.$$

The desired conclusion then follows from Lemma I.2.3 because $J\Psi(z) = CB_D(e,z)$ and $r\rho > m/2$. The proof of Lemma III.3 is complete, and Propositions III.2 and III.1 are thus entirely proved.

REMARK. In view of Proposition III.1, under the measure $B^{-r\rho/(-2d+q)}(z,z)\,dv(z)$ the Bergman kernel of D does not only reproduce functions in Bergman spaces A^p, but also those holomorphic functions in D, satisfying the uniform estimate (*). This property is trivial on bounded domains and seems to be new concerning unbounded domains; however, in this case, unlike the bounded domains, the meaning of the right-hand side of (7) requires the presence of a weight.

We have just proved that the Bergman projection P of D is bounded from L^∞ onto the Bloch space $\mathscr{B}$. Furthermore, if R denotes the operator defined in $\mathscr{B}$ with values in L^∞ that assigns to an element g of $\mathscr{B}$ the bounded function

$$Rg(z) = \mathscr{D}_{z_1}g(z)B^{-r\rho/(-2d+q)}(z,z), \qquad z=(z_1,z_2)\in D\subset \mathbf{C}^n\times\mathbf{C}^N,$$

R is a "continuous right inverse" of P, i.e., $PR = \mathrm{Id}_{\mathscr{B}}$.

The following theorem comes as a summary of this chapter:

THEOREM III.1. *Let D be a symmetric Siegel domain of type* II.

1° *The Bergman projection P of D is a bounded operator from L^∞ onto the Bloch space $\mathscr{B}$; furthermore, P possesses a continuous right inverse R: $\mathscr{B} \to L^\infty$;*

2° *Consequently, the dual of the Bergman space A^1 (the Bloch space $\mathscr{B}$) can be realized as the Bergman projection of L^∞.*

REMARKS. 1° At the end of the second chapter, we noticed that the dual of A^1 still coincides with the Bloch space in any homogeneous, but nonsymmetric Siegel domain of type II, associated with a self-conjugate cone. In such a domain, the Bergman projection can be defined exactly in the same way as an operator from L^∞ into $\mathscr{B}$. The question therefore arises whether Theorem III.1 can be extended to this type of domain; a relevant problem would be to find a substitute for Lemma III.2.

2° Unlike the particular cases of the Cayley transform of the unit ball and of the tube over the spherical cone, respectively studied in [**1** and **3**], we have not defined here the Bergman projection P of L^∞ as an integral operator associated with a kernel. The problem of determining a defining kernel for P will be considered in a forthcoming paper.

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